Quantum enhanced positioning and clock synchronization

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A wide variety of positioning and ranging procedures are based on repeatedly sending electromagnetic pulses through space and measuring their time of arrival. This paper shows that quantum entanglement and squeezing can be employed to overcome the classical power/bandwidth limits on these procedures, enhancing their accuracy. Frequency entangled pulses could be used to construct quantum positioning systems (QPS), to perform clock synchronization, or to do ranging (quantum radar): all of these techniques exhibit a similar enhancement compared with analogous protocols that use classical light. Quantum entanglement and squeezing have been exploited in the context of interferometry [1–5], frequency measurements [6], lithography [7], and algorithms [8]. Here, the problem of positioning a party (say Alice) with respect to a fixed array of reference points will be analyzed.

Alice's position may be obtained simply by sending pulses that originate from her position and measuring the time it takes for each pulse to reach the reference points. The time of flight, the speed of the pulses and the arrangement of the reference points determine her position. The accuracy of such a procedure depends on the number of pulses, their bandwidth and the number of photons per pulse. This paper shows that by measuring the correlations between the times of arrival of M pulses which are frequency-entangled, one can in principle increase the accuracy of such a positioning procedure by a factor $\sqrt{M}$ as compared to positioning using unentangled pulses with the same bandwidth. Moreover, if number-squeezed pulses can be produced [9], it is possible to obtain a further increase in accuracy of $\sqrt{N}$ by employing squeezed pulses of N quanta, *vs.* employing "classical" coherent states with N mean number of quanta. Combining entanglement with squeezing gives an overall enhancement of $\sqrt{MN}$. In addition, the procedure exhibits improved security: because the timing information resides in the entanglement between pulses, it is possible to implement [10] quantum cryptographic schemes that do not allow an eavesdropper to obtain information on the position of Alice. The primary drawbacks of this scheme are the difficulty of creating the requisite entanglement and the sensitivity to loss. On the other hand, the frequency entanglement allows similar schemes to be highly robust against pulse broadening due to transit through dispersive media [11].

The clock synchronization problem can be treated using the same method. In Refs. [12] and [13] two novel techniques for clock synchronization using entangled states are presented. However, the authors of Ref. [12] themselves point out that the resources needed by their scheme could be used to perform conventional clock synchronization without entanglement. Similarly, all the enhancement of [13] arises from employing high-frequency atoms which themselves could be used for clock synchronization to the same degree of accuracy without any entanglement. In neither case do these proposals give an obvious enhancement over classical procedures that use the same resources. Here, by contrast, it is shown that quantum features such as entanglement and squeezing can in principle be used to supply a significant enhancement of the accuracy of clock synchronization as compared to classical protocols using light of the same frequency and power. In fact, the clock synchronization can be accomplished by sending pulses back and forth between the parties whose clocks are to be synchronized and measuring the times of arrival of the pulses (Einstein's protocol). In this way synchronization may be treated on the same basis as positioning and the same accuracy enhancements may be achieved through entanglement and squeezing. In this paper only the positioning accuracy enhancement will be addressed in detail.

In order to introduce the formalism, the simple case of position measurement with classical coherent pulses is now presented. Since each dimension can be treated independently, the analysis will be limited to the one-dimensional case. For the sake of simplicity, consider the situation in which Alice wants to measure her position x by sending a pulse to each of M detectors placed in a known position (refer to Fig. 1). This can be easily generalized to different setups, such as the case in which the detectors are not all in the same location, the case in which only one detector is employed with M time-separated pulses, the case in which the pulses originate from the reference points and are measured by Alice (as in GPS), *etc.* Alice's estimate of her position is given by $x = c\frac{1}{M}\sum_{i=1}^{M} t_i$, where t_i is the travel time of the $i-$th pulse and c is the light speed. The variable t_i has an intrinsic indeterminacy dependent on the spectral characteristics and mean number of photons N of the i-th pulse. For example, given a Gaussian pulse of frequency spread $\Delta\omega$, according to the central limit theorem, t_i cannot be measured with an accuracy better than $1/(\Delta\omega\sqrt{N})$ since it is estimated at most from N data points (*i.e.* the times of arrival of the single photons, each having an indeterminacy $1/\Delta\omega$). Thus, if Alice uses M Gaussian pulses of equal frequency spread, the accuracy in the measurement of the average time of arrival is

$$\Delta t = \frac{1}{\Delta\omega\sqrt{MN}} \; . \tag{1}$$

Quantum Mechanics allows us to do much better. In order to demonstrate the gain in accuracy afforded by Quantum Mechanics, it is convenient to provide first a fully quantum analysis of the problem of determining the average time of arrival of a set of M classical pulses, each having mean number of photons N. Such a quantum treatment for a classical problem may seem like overkill, but once the quantum formalism is presented, the speed-up attainable in the fully quantum case can be derived directly. In addition, it is important to verify that no improvement over Eq. (1) is obtainable using classical pulses. The M coherent pulses are described by a state of the radiation field of the form

$$|\Psi\rangle_{cl} \equiv \bigotimes_{i=1}^{M} \bigotimes_{\omega} \left| \alpha(\phi_\omega\sqrt{N}) \right\rangle_i \; , \tag{2}$$

where ϕ_ω is the pulses' spectral function, $|\alpha(\lambda_\omega)\rangle_i$ is a coherent state of amplitude λ_ω in the mode at frequency ω directed towards the i-th detector, and N is the mean number of photons in each pulse. The pulse spectrum $|\phi_\omega|^2$ has been normalized so that $\int d\omega |\phi_\omega|^2 = 1$. For detectors with perfect time resolution, the joint probability for the i-th detector to detect N_i photons in the i-th pulse at times $t_{i,k}$ is given by [14]

$$p(\{t_{i,k}\}) \propto \left\langle : \prod_{i=1}^{M} \prod_{k=1}^{N_i} E_i^{(-)}(t_{i,k}) E_i^{(+)}(t_{i,k}) : \right\rangle \; , \tag{3}$$

where $t_{i,k}$ is the time of arrival of the k-th photon in the i-th pulse, shifted by the position of the detectors $t_{i,k} \to t_{i,k} + x/c$. The signal field at the position of the i-th detector at time t is given by $E_i^{(-)}(t) \equiv \int d\omega \; a_i^\dagger(\omega) \; e^{i\omega t}$ and $E_i^{(+)} \equiv \left(E_i^{(-)}\right)^\dagger$, where $a_i(\omega)$ is the field annihilator of a quantum of frequency ω at the i-th detector, which satisfies $[a_i(\omega), a_j^\dagger(\omega')] = \delta_{ij}\delta(\omega-\omega')$. The estimation of the ensemble average in Eq. (3) on the state $|\Psi\rangle_{cl}$, using the property $a(\omega') \otimes_\omega |\alpha(\lambda_\omega)\rangle = \lambda_{\omega'} \otimes_\omega |\alpha(\lambda_\omega)\rangle$, gives

$$p(\{t_{i,k}\}) \propto \prod_{i=1}^{M} \prod_{k=1}^{N_i} |g(t_{i,k})|^2 \; , \tag{4}$$

where $g(t)$ is the Fourier transform of the spectral function ϕ_ω. Averaging over the times of arrival $t_{i,k}$ and over the number of photons N_i detected in each pulse, one has

$$\langle t \rangle = \langle \frac{1}{M} \sum_{i=1}^{M} \frac{1}{N_i} \sum_{k=1}^{N_i} t_{i,k} \rangle = \overline{\tau} \;\; ; \;\; \Delta t \gtrsim \frac{\Delta\tau}{\sqrt{MN}} \; , \tag{5}$$

with approximate equality for $N \gg 1$. Here $\overline{\tau} \equiv \int dt \;\, t\,|g(t)|^2$ and $\Delta\tau^2 \equiv \int dt \;\, t^2 \;\, |g(t)|^2 - \overline{\tau}^2$ are independent of i and k since all the photons have the same spectrum. Eq. (5) is the generalization of (1) for non-Gaussian pulses.

Quantum light can exhibit phenomena that are not possible classically such as entanglement and squeezing, which, as will now be seen, can give significant enhancement for determining the average time of arrival. First consider entanglement. The framework just established allows the direct comparison between frequency entangled pulses and unentangled ones. For the sake of clarity, consider initially single photon entangled pulses.

Define the "frequency state" $|\omega\rangle$ for the electromagnetic field the state in which all modes are in the vacuum state, except for the mode at frequency ω which is populated by one photon. Thus the state $\int d\omega \; \phi_\omega |\omega\rangle$ represents a single photon wave packet with spectrum $|\phi_\omega|^2$. Consider the M-photon frequency entangled state given by

$$|\Psi\rangle_{en} \equiv \int d\omega \; \phi_\omega \; |\omega\rangle_1 |\omega\rangle_2 \cdots |\omega\rangle_M \; , \tag{6}$$

where the ket subscripts indicate the detector each photon is traveling to. Inserting $|\Psi\rangle_{en}$ in Eq. (3), and specializing to the case $N_i = 1$, it follows that

$$p(t_1, \cdots, t_M) \propto |g(\sum_{i=1}^{M} t_i)|^2 \; . \tag{7}$$

That is, the entanglement in frequency translates into the bunching of the times of arrival of the photons of different pulses: although their individual times of arrival are random, the average $t \equiv \frac{1}{M}\sum_{i=1}^{M} t_i$ of these times is highly peaked. (The measurement of t follows from the correlations in the times of arrival at the different detectors). Indeed, from Eq. (7) it results that the probability distribution of t is $|g\,(Mt)\,|^2$. This immediately implies that the average time of arrival is determined to an accuracy

$$\Delta t = \frac{\Delta\tau}{M} \; , \tag{8}$$

where $\Delta\tau$ is the same of Eq. (5). This result shows a $\sqrt{M}$ improvement over the classical case (5).

To emphasize the importance of entanglement, Eq. (8) should be compared to the result one would obtain from an unentangled state analogous to $|\Psi\rangle_{en}$. To this end, consider the state defined as

$$|\Psi\rangle_{un} \equiv \bigotimes_{i=1}^{M} \int d\omega_i \; \phi_{\omega_i} \; |\omega_i\rangle_i \; , \tag{9}$$

which describes M uncorrelated single photon pulses each with spectral function ϕ_ω. By looking at the spectrum of the state obtained by tracing away all but one of the modes in (6), each of the photons in (9) can be shown to have the same spectral characteristics as the photons in the entangled state $|\Psi\rangle_{en}$. Now, using Eq. (3) for the uncorrelated M photon pulses $|\Psi\rangle_{un}$, it follows that

$$p(t_1, \cdots, t_M) \propto \prod_{i=1}^{M} |g(t_i)|^2 \;, \tag{10}$$

which is the same result that was obtained for the classical state (2). Thus Eq. (5) holds, with $N = 1$, also for $|\Psi\rangle_{un}$. From the comparison of Eqs. (5) and (8), one sees that, employing frequency-entangled pulses, an accuracy increase by a factor $\sqrt{M}$ is obtained in the measurement of t with respect to the case of unentangled photons.

Since $|\Psi\rangle_{en}$ is tailored as to give the least indetermination in the quantity t, it is appropriate for the geometry of the case given in Fig. 1, where the sum of the time of arrival is needed. Other entangled states can be tailored for different geometric dispositions of the detectors, as will be shown through some examples.

How is it possible to create the needed entangled states? In the case $M = 2$, the twin beam state at the output of a cw pumped parametric downconverter will be shown to be fit. It is a 2 photon frequency entangled state of the form $\int d\omega\ \phi_\omega |\omega\rangle_s |\omega_0 - \omega\rangle_i$, where ω_0 is the pump frequency and s and i refer to the signal and idler modes respectively. This state is similar to (6) and it can be employed for position measurements when the two reference points are in opposite directions, *e.g.* one to the left and one to the right of Alice. In fact, it can be seen that $p(t_1, t_2) \propto |g(t_1 - t_2)|^2$ and hence such a state is optimized for time of arrival difference measurements, as experimentally reported in [15]. In the case of $M = 3$, a suitable state can be obtained starting from a 3-photon generation process that creates a state of the form $\int d\omega d\omega'\ f(\omega, \omega')|\omega\rangle|\omega'\rangle|\omega_0 - \omega - \omega'\rangle$, and then performing a non-demolition (or a post-selection) measurement of the frequency difference of two of the photons. This would create a maximally entangled 3-photon state, tailored for the case in which Alice has one detector on one side and two detectors on the other side. However, for $M > 2$, the creation of such frequency-entangled states represents a continuous variable generalization of the GHZ state, and, as such, is quite an experimental challenge.

Now turn to the use of number-squeezed states to enhance positioning. The N-th excitation of a quantum system (*i.e.* the state $|N\rangle$ of exactly N quanta) has a de Broglie frequency N times the fundamental frequency of the state. Its shorter wavelength makes such a state appealing for positioning protocols. In this case, the needed "frequency state" is given by $|N_\omega\rangle$, defined as the state where all modes are in the vacuum except for the mode at frequency ω, which is in the Fock state $|N\rangle$. The probability of measurement of N quanta in a single pulse at times $t_1, \cdots, t_N$ is given by Eq. (3) with $M = 1$ detectors. It is straightforward to see that, for a state of the form $\int d\omega\ \phi_\omega\ |N_\omega\rangle$, the time of arrival probability is given by

$$p(t_1, \cdots, t_N) \propto |g(\sum_{k=1}^{N} t_k)|^2 \;. \tag{11}$$

Such a result must be compared to what one would obtain employing a classical pulse $|\Psi\rangle_{cl}$ of N mean number of photons, *i.e.* the state (2) with $M = 1$. Its probability (4) shows that employing the N-photon Fock state gives an accuracy increase of $\sqrt{N}$ *vs.* the coherent state with N mean number of photons. The similarity of this result (11) with the one obtained in Eq. (7) stems from the fact that the Fock state $|N_\omega\rangle$ can be interpreted as composed by N one-photon pulses of identical frequency. Hence, all the results and considerations obtained previously apply here. An experiment which involves such a state for $N = 2$ is reported in [16].

Entangled pulses of number-squeezed states combine both these enhancements. By replacing $|\omega\rangle$ with the number-squeezed states $|N_\omega\rangle$ in the M-fold entangled state (6), one immediately obtains an improvement of $\sqrt{MN}$ over the accuracy obtainable by using M classical pulses of N photons each.

The enhanced accuracy achieved comes at the cost of an enhanced sensitivity to loss. If one or more of the photons fails to arrive, the time of arrival of the remaining photons do not convey any timing information. The simplest way to solve this problem is to ignore all trials where one or more photons is lost. A more sophisticated method is to use *partially* entangled states: these states provide a lower level of accuracy than fully entangled states, but are more tolerant to loss. As shown in figure 2, even the simple protocol of ignoring trials with loss still surpasses the unentangled state accuracy limit even for significant loss levels. The use of intrinsically loss-tolerant, partially entangled states does even better [10].

Before closing, it is useful to consider the following intuitive picture of quantum measurements of timing. A quantum system such as a pulse of photons or a measuring apparatus with spread in energy ΔE can evolve from one state to an orthogonal state in time Δt no less than $\pi\hbar/(2\Delta E)$ [17]. Accordingly, to make more accurate timing measurements, one requires states with sharp time dependence, and hence high spreads in energy. Classically, combining M systems each with spread in energy ΔE results in a joint system with spread in energy $\sqrt{M}\Delta E$. Quantum-mechanically, however, M systems can be put in entangled states in which the spread in energy is proportional to $M\Delta E$. Similarly, N photons can be joined in a squeezed state with spread in energy $N\Delta E$. The Margolus-Levitin theorem [18] limits the time Δt it takes for a quantum system to evolve from one state to an orthogonal one by $\Delta t \geq 2\hbar/\pi E$, where E is the average energy of a system (taking the ground state energy to be 0). This result implies that the $\sqrt{MN}$ enhancement presented here is the best one can do.

In conclusion, quantum entanglement and squeezing have been shown to increase the accuracy of position measurements, and, as a consequence, they can also be employed to improve the accuracy in distant clock synchronization. For maximally entangled M-particle states

we have shown an accuracy increase $\propto \sqrt{M}$ *vs.* unentangled states with identical spectral characteristics. A further increase $\propto \sqrt{N}$ in accuracy in comparison with classical pulses was also shown for the measurement of N quanta states. At least for the simple cases of $M = 2$ or $N = 2$, the described protocols are realizable in practice.

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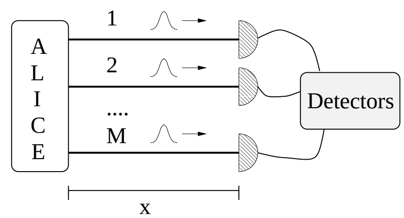

FIG. 1. Sketch of the idealized experimental configuration. Alice sends M light pulses to the M detectors. She averages the times of arrival t_i of the pulses to recover her unknown position x.

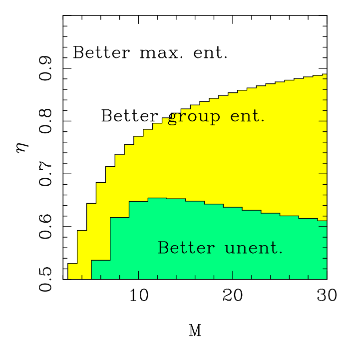

FIG. 2. Sensitivity to loss. The quantum efficiency η needed for having an accuracy increase over the unentangled state $|\Psi\rangle_{un}$ is plotted *vs.* the number M of photons (here $N = 1$). The upper white region is where $|\Psi\rangle_{en}$ does better than $|\Psi\rangle_{un}$. The white and light grey regions are where a partially entangled state, which exploits a configuration where one partially entangles subgroups of 2 maximally entangled photons, does better than $|\Psi\rangle_{un}$. The lower dark region is where $|\Psi\rangle_{un}$ does better.